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Theoretical model of discrete tone generation by impinging jets

By CHRISTOPHER K. W. TAM[1] AND K. K. AHUJA[2]
[1] Department of Mathematics, Florida State University, Tallahassee, FL 32306-3027, USA
[2] LASC–Georgia, Marietta, GA 30063, USA

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It is well known that when a high subsonic (Mach number > 0.7) high Reynolds number ($Re > 2\times10^5$) jet is directed normal to a wall intense discrete frequency sound waves called impingement tones are emitted. This phenomenon has been studied by a number of investigators in the past. It is generally accepted that the tones are generated by a feedback loop. Despite this general agreement critical difference in opinion as to how the feedback is achieved remains unresolved. Early investigators (e.g. Wagner 1971; Neuwerth 1973, 1974) proposed that the feedback loop is closed by acoustic disturbances which propagate from the wall to the nozzle exit inside the jet. Recent investigators (e.g. Ho & Nosseir 1981; Umeda *et al.* 1987), However, believed that the feedback is achieved by sound waves propagating outside the jet. In this paper a new feedback mechanism is proposed. It is suggested that the feedback is achieved by upstream-propagating waves associated with the lowest-order intrinsic neutral wave modes of the jet flow. These wave modes have well-defined radial and azimuthal pressure and velocity distributions. These distributions are dictated by the mean flow of the jet exactly as in the case of the well-known Kelvin–Helmholtz instability waves. The characteristics of these waves are calculated and studied. These characteristics provide a natural explanation of why the unsteady flow fields of subsonic impinging jets must be axisymmetric, whereas those for supersonic jets may be either axisymmetric or helical (flapping). In addition they also offer, for the first time, an explanation as to why no stable impingement tones have been observed for (cold) subsonic jets with Mach number less than 0.6. Furthermore, the new model allows the prediction of the average Strouhal number of impingement tones as a function of jet Mach number. The predicted results compare very favourably with measurements. For subsonic jets the pressure and velocity field of these upstream-propagating neutral waves are found to be confined primarily inside the jet. This is in agreement with the observations of Wagner (1971) and Neuwerth (1973, 1974) and their contention that the feedback disturbances actually propagate upstream inside the jet.

1. Introduction

It was recognized by early investigators e.g. Wagner (1971) and Neuwerth (1973, 1974) that very strong discrete tones were generated when a high subsonic (Mach number > 0.7) or supersonic jet was directed normal on a wall at a distance of several jet diameter away. By varying the distance between the wall and the nozzle exit it was observed that the tone frequency exhibited a staging phenomenon suggesting that the tones were generated by a feedback loop. The feedback loop derives its

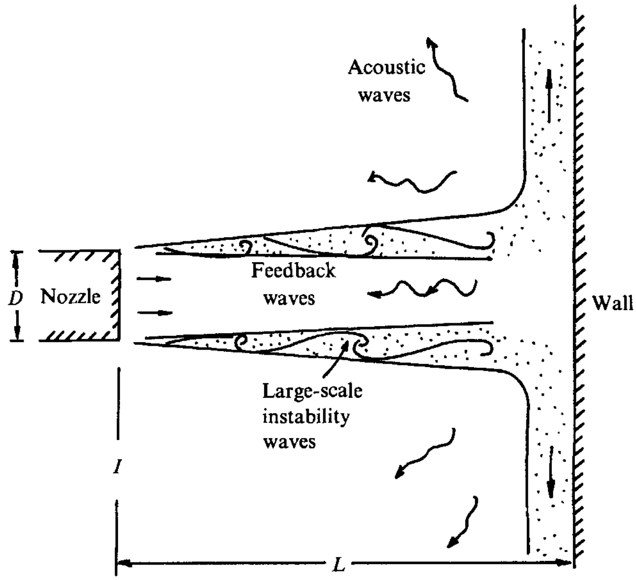


FIGURE 1. Schematic diagram of a subsonic impinging jet and possible feedback paths.

energy from the large-scale instability waves in the mixing layer of the jet. These waves are generated near the nozzle lip. They propagate downstream in the mixing layer of the jet as shown in figure 1. Upon impinging on the wall strong acoustic waves are produced. The acoustic waves propagate upstream to the nozzle exit region of the jet where they excite the shear layer. This excitation causes further generation of instability waves and thus completes the feedback cycle. In his pioneering work on subsonic impinging jets Wagner (1971) carried out extensive schlieren study of the flow field inside and outside the jet. He reported the observation of a standing wave pattern inside the jet which he proposed was formed by the superposition of the pressure fields of the downstream-propagating instability wave and the upstream-propagating feedback acoustic waves. Neuwerth (1973) improved the experimental facility of Wagner. By observing the feedback phenomenon using high-speed movies he was able to identify clearly the upstream feedback waves inside the jet column. Recently Ho & Nosseir (1981) and Umeda, Maeda & Ishii (1987) performed detailed investigations of the same feedback cycle associated with the impingement tones at high subsonic Mach numbers. Based on their extensive correlation measurements they concluded, in total disagreement with Wagner and Neuwerth, that the feedback acoustic waves actually propagated and closed the loop outside the jet column. The disagreement is extremely serious and irreconcilable. It calls for a fundamental re-examination of how the acoustic feedback is really accomplished.

Since the start of the present investigation a careful study of all available schlieren and shadowgraphic pictures of resonant impinging jets including those obtained by Wagner, Neuwerth, Umeda *et al.* and the present data (see figure 2) was carried out. The study reveals that for subsonic jets the instability waves and the flow field associated with the feedback loop invariably possess axial symmetry. It is well known that round jets can undergo axisymmetric as well as helical and flapping mode instability (higher-order modes are not often observed). Neuwerth (1981) observed axisymmetric coherent structures (instability waves) in a free jet at a

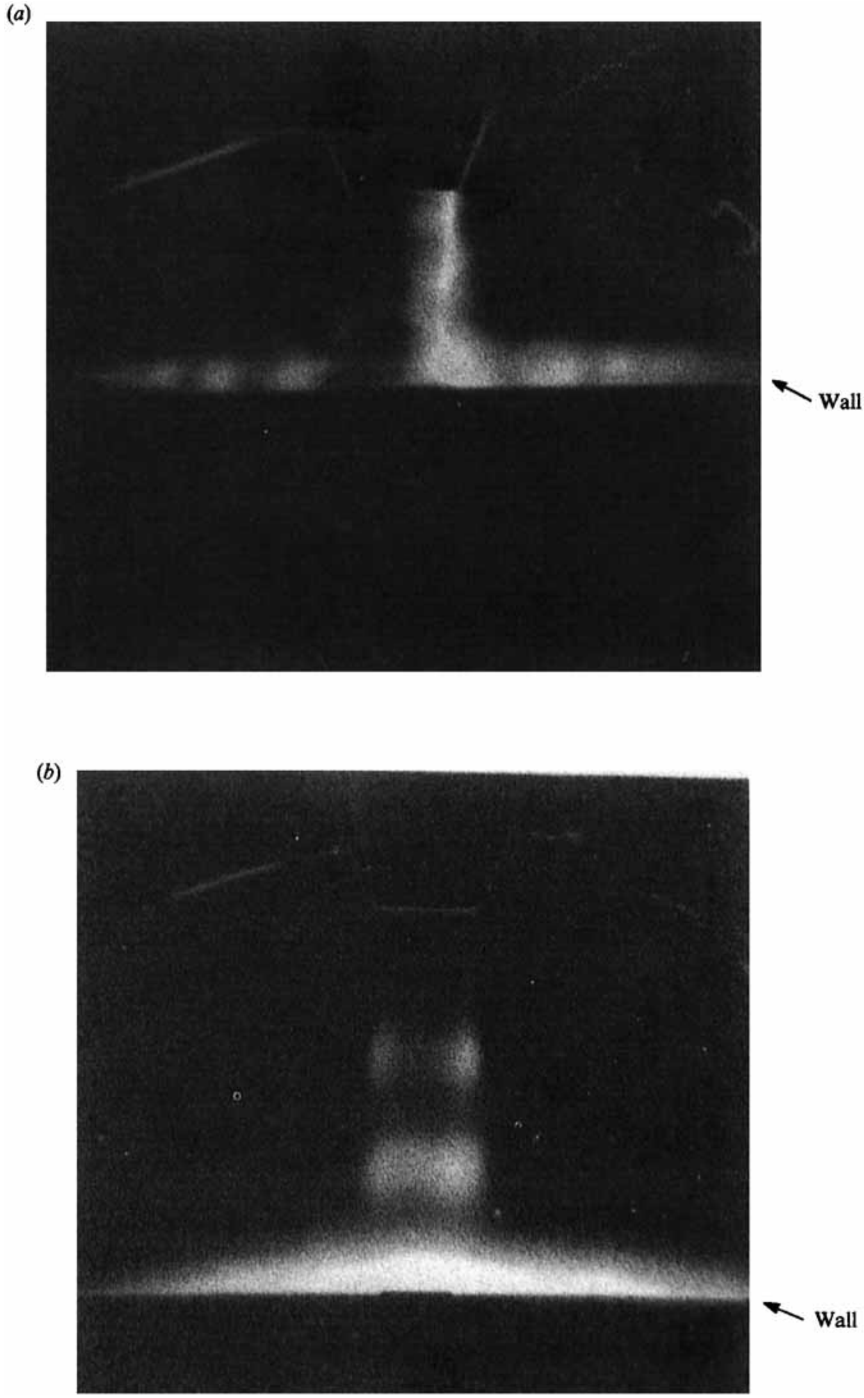
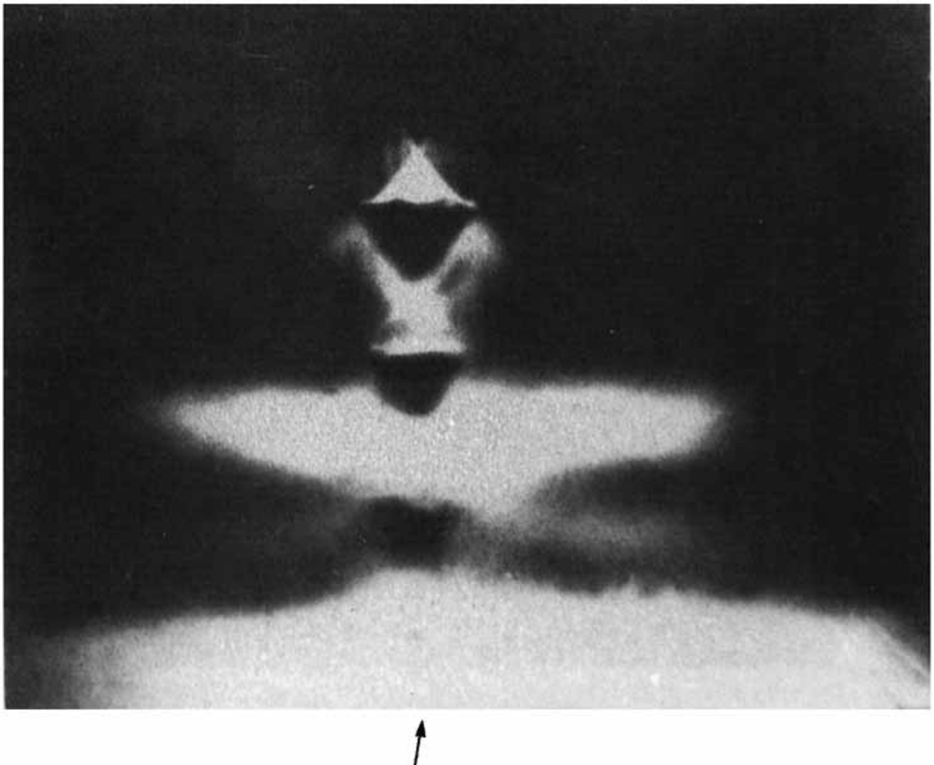


FIGURE 2. Subsonic impinging jet at Mach number 0.8 showing axisymmetric large-scale instability waves in the jet column. (*a*) $L/D = 2.0$, (*b*) $L/D = 4.0$.



Wall

FIGURE 3. Supersonic impinging jet at Mach number 1.4 showing axisymmetric feedback resonance. $L/D = 6.0$.

Reynolds number of 10^6, and Mach number of 0.5 but a helical mode appeared when the Mach number was increased beyond 0.8. However, when the impingement plate was inserted only the axisymmetric instability mode remained. Nosseir & Ho (1982) measured the azimuthal pressure cross-correlation around resonant impinging jets at Mach numbers > 0.7. The correlation at different azimuthal angles was sinusoidal with respect to the time delay τ. The correlation function has a maximum at $\tau = 0$ indicating that the impinging coherent structures (instability waves) and the feedback acoustic waves are predominantly axisymmetric. Based on all the above data it appears that only the axisymmetric instability and acoustic wave modes are allowed in subsonic resonant impinging jets. This is in sharp contrast to supersonic impinging jets. Earlier, Neuwerth (1974) had observed both axisymmetric and helical (or flapping) resonance modes in these jets. These observations at supersonic Mach numbers are confirmed by the present experiments (see figures 3 and 4). As yet there seems to be no theoretical explanation of this Mach number influence on the helical or flapping resonance mode, namely, that they are not observed in subsonic impinging jets. A practical implication of this phenomenon is that twin subsonic impinging jets are less likely to undergo coupled synchronized oscillations, the reason being that twin jet resonance requires the jets to perform coordinated flapping oscillations (see Seiner, Manning & Ponton 1986, 1987; Norum & Shearin 1986). The flapping motion is equivalent to a superposition of equal amount of left-hand and right-hand helical mode oscillations. Since flapping is not allowed for single subsonic

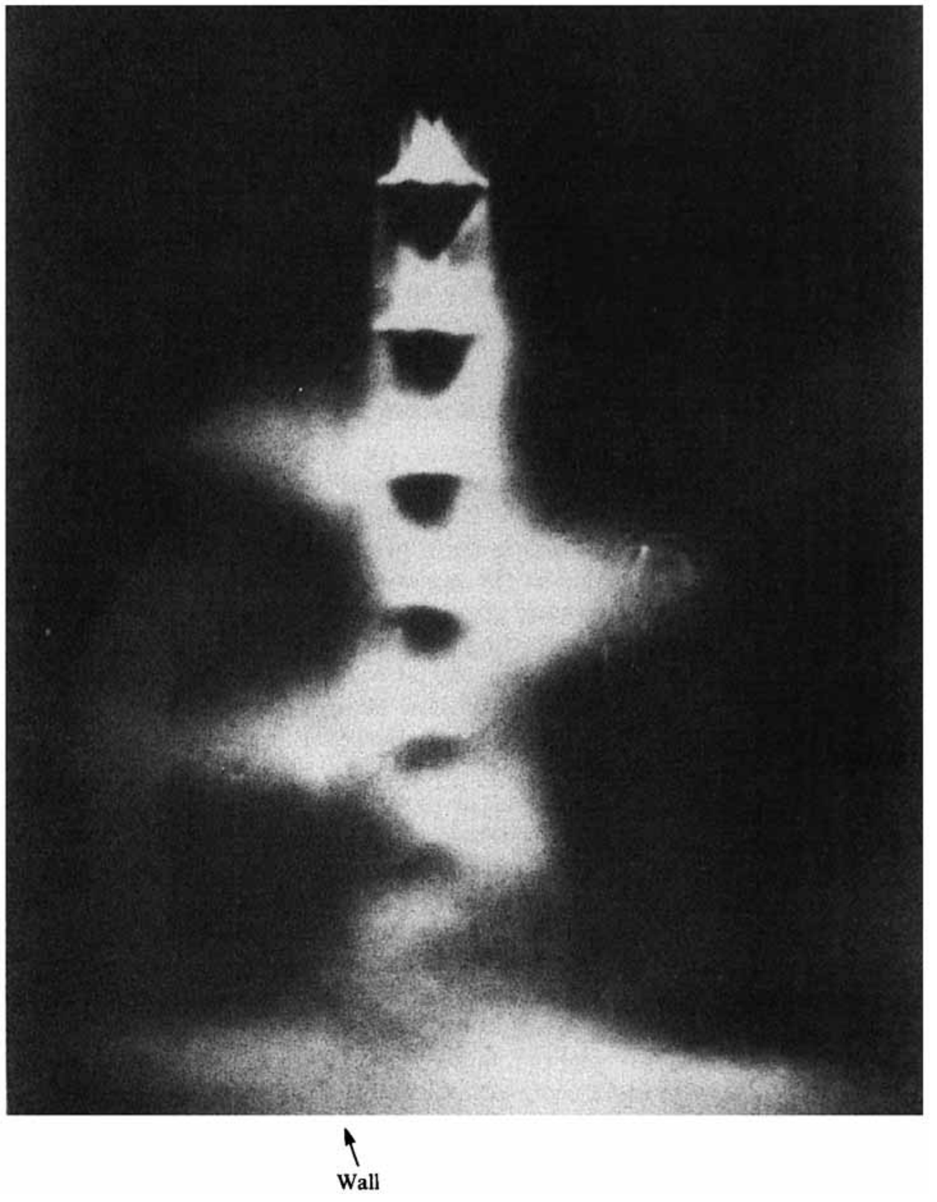


FIGURE 4. Supersonic impinging jet at Mach number 1.4 showing helical feedback resonance. $L/D = 10.0$.

impinging jets, it is, therefore, less probable that they will interact with each other to produce twin impinging jet resonance.

Another puzzling observation of the impingement tone phenomenon is the existence of a low Mach number cut-off for stable tones. It was first observed by Wagner (1971) that as the Mach number of the jet decreases below 0.7 stable impingement tones are difficult to maintain. At still lower Mach numbers, say below 0.6, no stable tones could be detected. This cut-off phenomenon was also observed and reported by Neuwerth (1974) and Ho & Nosseir (1981). So far no theoretical explanation of this Mach number effect has been offered in the literature.

The main objective of the present work is to develop a new analytical model and better understanding of the feedback loop. This model is capable of clarifying:

(*a*) whether the feedback acoustic waves propagate inside or outside a subsonic impinging jet in closing the self-excited resonance loop;

(*b*) why the feedback resonance in subsonic impinging jets must be associated with the axisymmetric mode alone whereas both helical (flapping) and axisymmetric modes are possible in supersonic jets;

(*c*) why there is a low Mach number cut-off for the existence of impingement tones.

The present effort, both theoretical and experimental, concentrates primarily on the feedback phenomenon. Its relationship to noise generation, both discrete and broadband, will not be dealt with in this paper.

2. Characteristics of feedback acoustic wave modes

As pointed out above, previous investigators, e.g. Wagner (1971), Neuwerth (1973, 1974), Ho & Nosseir (1981), Umeda *et al.* (1987) all believe that the impingement tones are generated by a feedback loop. The energy of the feedback loop is provided by the instability waves in the mixing layer of the jet. The instability waves are generated by acoustic excitations in the region near the nozzle exit. These waves grow as they propagate downstream, see figure 1. Upon impingement on the wall acoustic waves are generated. According to Wagner and Neuwerth the feedback acoustic waves travelled upstream inside the jet column for subsonic impinging jets. On the other hand, Ho & Nosseir suggested that the feedback acoustic waves actually travelled upstream outside the jet. However, regardless of whether the feedback waves are inside or outside the jet column these waves, on reaching the nozzle exit, tend to excite the shear layer of the jet, leading to the generation of instability waves. In this way the feedback loop is closed.

Let L be the distance between the wall and the nozzle exit and C_{i} and C_{a} be the phase velocities of the downstream propagating instability waves and the feedback acoustic waves respectively as shown in figure 1. The impingement tone frequency, f_n, is determined by the feedback condition that the time taken for the instability wave to propagate from the nozzle exit to the wall plus the time taken for the feedback acoustic waves to propagate from the wall upstream inside or outside the jet to the nozzle exit must be equal to an integral multiple of the period of oscillation. This condition leads to the following formula for f_n (Neuwerth 1974):

$$f_n = \frac{n}{L([1/C_{\mathrm{i}}]+[1/C_{\mathrm{a}}])}; \quad n = 1, 2, 3, \tag{1}$$

In deriving (1) it has tacitly been assumed that a negligibly small amount of time is involved in the reflection and the excitation processes of the feedback loop at the wall and at the nozzle exit. In this equation n is an arbitrary integer. The staging phenomenon alluded to before is the result of an abrupt change in the value of n as the nozzle to wall distance L increases. This gives rise to a discontinuous change in the impingement tone frequency versus L plot as shown in figure 15.

In the above feedback model, the downstream link of the loop, namely the instability waves, is well defined. The properties of these waves can be calculated with sufficient accuracy by the hydrodynamic instability wave theory (see for example, Michalke 1984; Tam & Burton 1984). The feedback acoustic waves meanwhile are somewhat unclear. In the work of Wagner, an attempt was made to account for the reflection and refraction of these waves by the jet mixing layer which, for convenience, was approximated by a plane vortex sheet. However, Wagner's model has not yielded results that agreed quantitatively with experiment. Nor has his model been able to predict the Strouhal number of the impingement tones. In the Ho & Nosseir model, the feedback waves were assumed to be acoustic waves with no particular spatial mode structure or properties. In this work it is our intention to express the belief that the Wagner model and the Ho & Nosseir model are either too simple or incorrect. We would like to suggest an alternative proposal that the feedback is achieved by waves belonging to the intrinsic upstream-propagating neutral acoustic modes of the jet flow. These upstream-propagating acoustic wave modes, just as the instability wave modes, have well-defined radial and azimuthal structures. Also they are, as in the case of Kelvin–Helmholtz instability waves, supported and determined by the mean flow of the jet. The characteristics of these waves will be calculated and discussed below.

In the present modified feedback model it is assumed that upon impinging on the wall (see figure 1) the instability waves excite the waves of the upstream-propagating neutral acoustic modes of the jet. These waves propagate toward the nozzle exit guided by the jet. Upon reaching the nozzle lip region they excite the instability waves of the jet and thus close the feedback loop. According to this new model, the frequency and characteristics of the feedback cycle are dictated by the intrinsic characteristics of the upstream-propagating neutral waves and those of the instability waves. It will be shown later that this is consistent with observations and that by calculating the properties of the upstream-propagating neutral waves, the properties of the impingement tones can be predicted.

2.1. *Vortex-sheet jet model*

To determine the propagation characteristics of the upstream-propagating neutral acoustic wave modes of a jet we shall model the jet as a uniform stream bounded by a vortex sheet. Let us consider small-amplitude disturbances superimposed on such a vortex sheet jet of velocity U_j and radius R_j as shown in figure 5. Let p_+ and p_- be the pressures associated with the disturbances outside and inside the jet and $\zeta(x,\theta,t)$ be the radial displacement of the vortex sheet, where (r,x,θ) are the cylindrical coordinates. By starting from the linearized equation of motion of a compressible flow, it is straightforward (see e.g. Tam 1972; Chan & Westley 1973; Tam & Hu 1989) to show that the governing equations and boundary conditions for p_+, p_-, and ζ are

$$\frac{1}{a_\infty^2}\frac{\partial p_+^2}{\partial t^2}-\nabla^2 p_+ = 0, \quad r \geqslant R_j, \tag{2a}$$

$$\frac{1}{a_j^2}\left(\frac{\partial}{\partial t}+U_j\frac{\partial}{\partial x}\right)^2 p_- - \nabla^2 p_- = 0, \quad r \leqslant R_j. \tag{2b}$$

At $r = R_j$,

$$p_+ = p_-, \tag{3}$$

$$\frac{\partial^2 \zeta}{\partial t^2} = -\frac{1}{\rho_\infty}\frac{\partial p_+}{\partial r}, \tag{4}$$

$$\left(\frac{\partial}{\partial t}+U_j\frac{\partial}{\partial x}\right)^2 \zeta = -\frac{1}{\rho_j}\frac{\partial p_-}{\partial r}. \tag{5}$$

At $r \rightarrow \infty$, p_+ satisfies the outgoing wave or boundedness condition, (6)

where a_∞, a_j, ρ_∞ and ρ_j are the speed of sound and gas density outside and inside the jet.

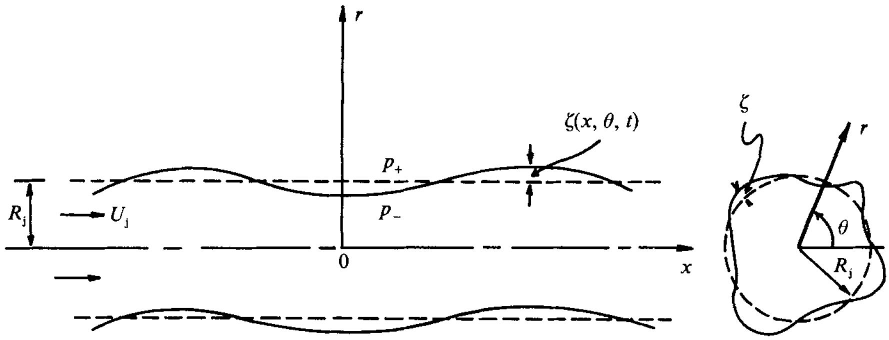


FIGURE 5. Small-amplitude disturbances superimposed on a vortex sheet jet.

Now let us look for propagating wave solutions of the form

$$\begin{bmatrix} p_+(r,x,\theta,t) \\ p_-(r,x,\theta,t) \\ \zeta(x,\theta,t) \end{bmatrix} = \begin{bmatrix} \hat{p}_+(r) \\ \hat{p}_-(r) \\ \hat{\zeta} \end{bmatrix} e^{i(kx+n\theta-\omega t)}, \tag{7}$$

where $n = 0, \pm 1, \pm 2, \ldots$ and k, the wavenumber and $\omega(\omega > 0)$, the angular frequency are as yet unspecified parameters. Substitution of (7) into (2)–(6) and eliminating $\hat{\zeta}$, it is easy to find that $\hat{p}_+$ and $\hat{p}_-$ are given by the solution of the following eigenvalue problem:

$$\frac{d^2\hat{p}_+}{dr^2} + \frac{1}{r}\frac{d\hat{p}_+}{dr} - \frac{n^2}{r^2}\hat{p}_+ + \left[\frac{\omega^2}{a_\infty^2} - k^2\right]\hat{p}_+ = 0, \tag{8}$$

$$\frac{d^2\hat{p}_-}{dr^2} + \frac{1}{r}\frac{d\hat{p}_-}{dr} - \frac{n^2}{r^2}\hat{p}_- + \left[\frac{(\omega - U_j k)^2}{a_j^2} - k^2\right]\hat{p}_- = 0. \tag{9}$$

At $r = R_j$,

$$\hat{p}_+ = \hat{p}_-, \tag{10}$$

$$\frac{1}{\rho_\infty \omega^2}\frac{d\hat{p}_+}{dr} = \frac{1}{\rho_j(\omega - U_j k)^2}\frac{d\hat{p}_-}{dr}. \tag{11}$$

The solution of (8) and radiation or boundedness condition (6) is

$$\hat{p}_+ = A H_n^{(1)}(\eta_+ r), \tag{12}$$

where $\eta_+ = (\omega^2/a_\infty^2 - k^2)^{\frac{1}{2}}$; $0 \leqslant \arg \eta_+ \leqslant \pi$, the left (right)-hand equality sign is to be used if ω is real and $\omega > 0$ $(\omega < 0)$. $H_n^{(1)}(\;)$ is the nth-order Hankel function of the first kind.

The solution of (9) which is bounded as $r \to 0$ is

$$\hat{p}_- = J_n(\eta_- r), \tag{13}$$

where $\eta_- = [(\omega - U_j k)^2/a_j^2 - k^2]^{\frac{1}{2}}$; $0 \leqslant \arg \eta_- < \pi$. By substituting (12) and (13) into (10), the unknown constant A can be easily found. This gives

$$\hat{p}_+ = \frac{J_n(\eta_- R_j)}{H_n^{(1)}(\eta_+ R_j)} H_n^{(1)}(\eta_+ r). \tag{14}$$

Upon imposing boundary condition (11) on (13) and (14), the condition of non-trivial solution leads to the following dispersion relation:

$$\eta_+ J_n(\eta_- R_j)\frac{H_n^{(1)\prime}(\eta_+ R_j)}{H_n^{(1)}(\eta_+ R_j)} - \frac{a_j^2}{a_\infty^2}\frac{C^2}{(C - M_j a_j/a_\infty)^2}\eta_- J_n'(\eta_- R_j) = 0, \tag{15}$$

where $C = \omega/(ka_\infty)$ and $M_j = U_j/a_j$ are the dimensionless phase velocity of the wave and Mach number of the jet respectively. A prime indicates the derivative.

The dispersion relation (15) has been used successfully by various investigators in the past, (e.g. Tam 1972; Chan & Westley 1973; Tam & Hu 1989), to calculate the properties of the Kelvin–Helmholtz instability waves of the jet flow. However, this dispersion relation as formulated is completely general and should describe all other possible intrinsic wave modes of the jet as well. Recently Tam & Hu (1989) carried out an in-depth investigation of the wave modes of high-speed compressible jets using dispersion relation (15). They found two additional families of wave solutions beside the familiar Kelvin–Helmholtz instability waves. For subsonic jets, although only a casual study was carried out, they noticed that the jet can support a family of neutral wave modes. These waves are neither unstable nor damped. That is, both k and ω of these waves are real. These waves propagate upstream along the jet column and possess properties in many ways resembling those of acoustic modes inside a circular hard-wall duct. These neutral wave modes are formed by reflections of the pressure waves at the mixing layer of the jet. We believe that these waves are generated in high subsonic impinging jets by the impingement of the Kelvin–Helmholtz instability waves on the wall. They were the upstream-propagating waves observed by Wagner (1971), Neuwerth (1974) and Umeda *et al.* (1987). These waves provide the dominant feedback from the wall to the nozzle lip region of the jet. In the next subsection the characteristics of these upstream-propagating neutral waves are analysed.

2.2. *Modal classification and wavenumber–frequency relations*

The upstream-propagating waves that are of primary concern here are neutral waves. For neutral waves, both the wavenumber k and the angular frequency ω are real. For the purpose of computing the dispersion curve, $\omega = \omega(k)$, it is advantageous to rewrite dispersion relation (15) and eigenfunction (13) and (14) each in a form involving only real functions. This can be done by using the standard relation between Hankel function $H_n^{(1)}$ with imaginary argument and modified Bessel function K_n:

$$|\xi_+|\, J_n(|\xi_-\alpha|)\frac{[K_{n-1}(|\xi_+\alpha|)+K_{n+1}(|\xi_+\alpha|)]}{K_n(|\xi_+\alpha|)}+\frac{C^2|\xi_-|}{(a_\infty C/a_j-M_j)^2}[J_{n-1}(|\xi_-\alpha|) - J_{n+1}(|\xi_-\alpha|)] = 0, \quad \omega > 0, \tag{16}$$

$$\hat{p}_+ = \frac{J_n(|\xi_-\alpha|)}{K_n(|\xi_+\alpha|)}K_n\left(\frac{|\xi_+\alpha|\, r}{R_j}\right), \tag{17}$$

$$\hat{p}_- = J_n\left(\frac{|\xi_-\alpha|\, r}{R_j}\right), \tag{18}$$

where $\xi_+ = |C^2-1|^{\frac{1}{2}}$, $\xi_- = |(a_\infty C/a_j - M_j)^2-1|^{\frac{1}{2}}$ and $\alpha = kR_j$. The upstream-propagating neutral wave modes are given by the roots of (16) along the negative real axis of the complex C-plane lying between the branch points of η_+ or ξ_+ at $C = -1$ and η_- or ξ_- at $C = (a_j/a_\infty)(M_j-1)$.

For given values of α and n, the dispersion relation (16) has many roots or eigenvalues. The eigenfunction corresponding to each eigenvalue is given by (17) and (18). The entire set of eigenvalues can be characterized by the azimuthal wavenumber n and a radial wavenumber m. For convenience, the wave mode which corresponds to the nth azimuthal mode and the mth radial mode will be designated by (n, m), where $n = 0, 1, 2, \ldots$ and $m = 1, 2, 3, \ldots$. Figure 6 shows typical eigenfunction

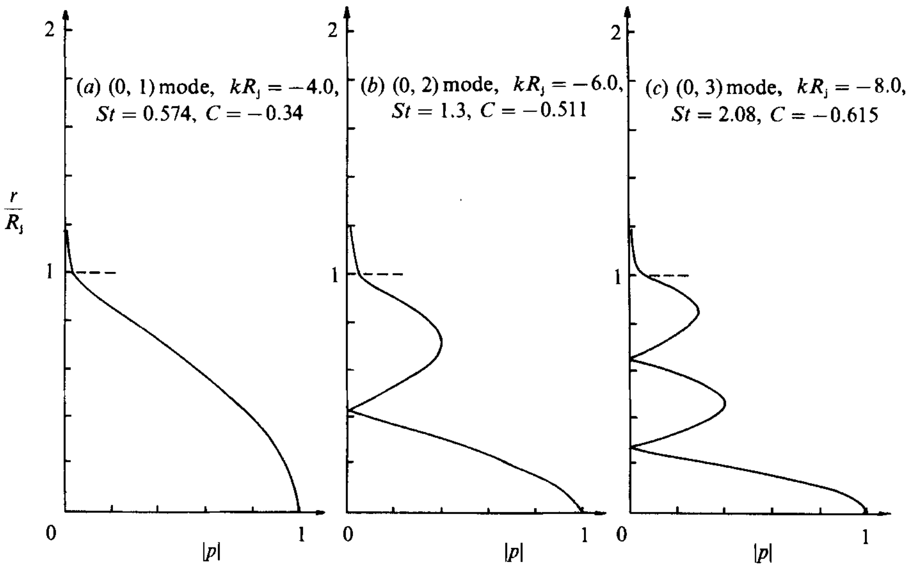


FIGURE 6. Eigenfunction distribution of axisymmetric neutral wave modes for cold jets; $M_j = 0.8$. Strouhal number $(St) = 2fR_j/U_j$.

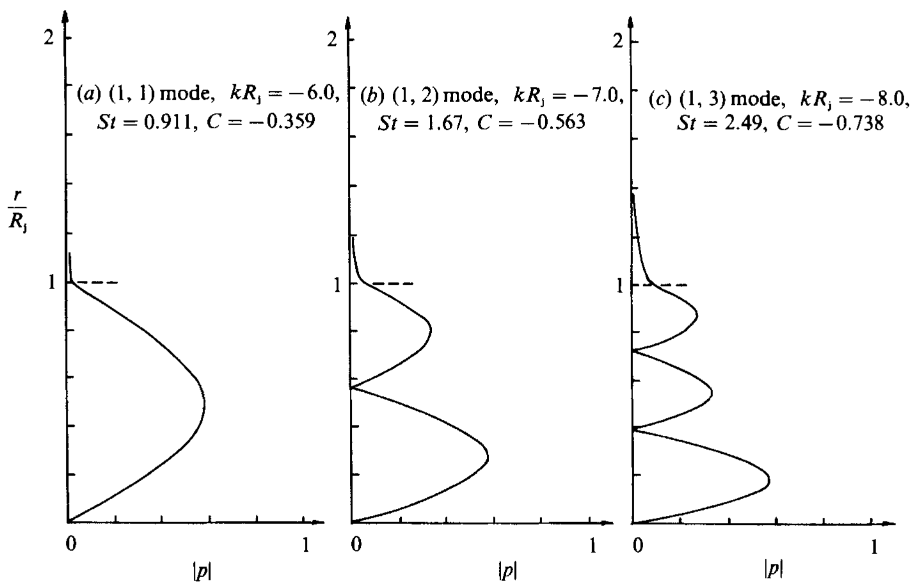


FIGURE 7. Eigenfunction distribution of helical neutral wave modes for cold jets: $M_j = 0.8$.

distributions for cold jets (the total temperature of the jets is equal to the ambient temperature) at $M_j = 0.8$ for the (0, 1), (0, 2) and (0, 3) modes at different Strouhal number ($St = fD/U_j$, where f is the frequency and D is the nozzle exit diameter). The radial index m characterizes the number of maxima of $|p|$. For the $n = 0$ modes, $|p|$ attains its maximum value at $r = 0$. Figure 7 shows typical eigenfunction distributions at $M_j = 0.8$ for the (1, 1), (1, 2) and (1, 3) modes. For wave modes with $n = 1$ or higher, the eigenfunction is zero at the jet axis. It is to be noted that unless

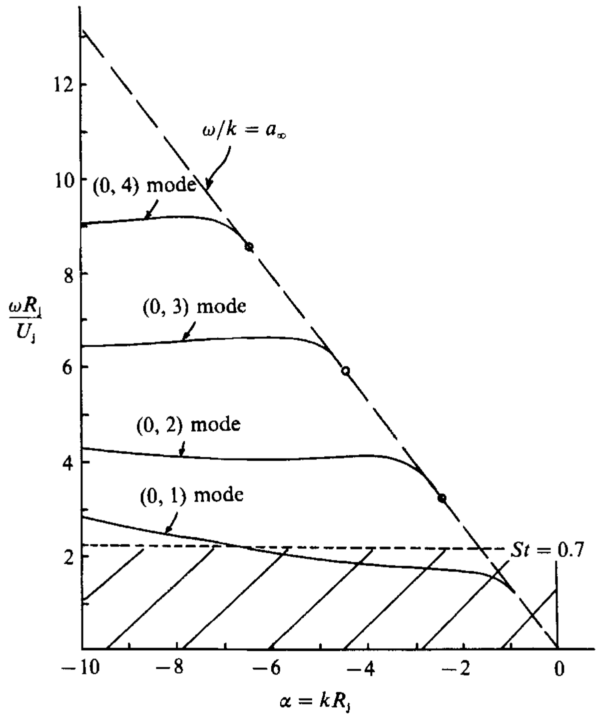


FIGURE 8. Wavenumber–frequency relations of axisymmetric neutral waves for cold jets; $M_j = 0.8$. The frequency range for which the Strouhal number ($St = fD/U_j$) of the waves is less than 0.7 is hatched. Only waves in this region have frequency which can match that of the Kelvin–Helmholtz instability waves of the jet flow.

the C-value of the wave is close to -1, the pressure fluctuation of the wave is almost entirely confined to regions inside the jet flow.

The wavenumber–frequency relationships of the axisymmetric modes (0, 1), (0, 2), (0, 3) and (0, 4) for the $M_j = 0.8$ (cold) jets are shown in figure 8. These are typical dispersion relations at subsonic Mach numbers. As the dimensionless wavenumber $\alpha = kR_j$ increases, the dispersion relation of each mode terminates along the straight line $C = -1$. In figure 8 the cut-off points are indicated by small circles. Figure 9 shows the corresponding dispersion relations for the helical modes (1, 1), (1, 2) and (1, 3) at $M_j = 0.8$. These are typical of all the higher-order modes in the index n. Again the dispersion relations terminate on the line $C = -1$ as indicated.

For supersonic jets, similar neutral waves exist. Figures 10 and 11 are the wavenumber–frequency relations of these wave modes. They are typical for supersonic jets. Again all the dispersion relations terminate or have cut-off points along the line $C = -1$. In the work of Tam & Hu (1989) a detailed investigation of these neutral waves in supersonic jets was carried out in the complex-k and complex-ω planes. They have shown that for a specified mode, the upstream-propagating neutral waves are given by the part of the dispersion curve lying to the right of the maximum point where the slope (or group velocity) is negative. These waves have a phase velocity nearly equal to the ambient speed of sound, i.e. $C \approx -1$. To the left of the maximum point the waves are downstream propagating. Figures 12 and 13 show the eigenfunction distributions of the first three axisymmetric and helical

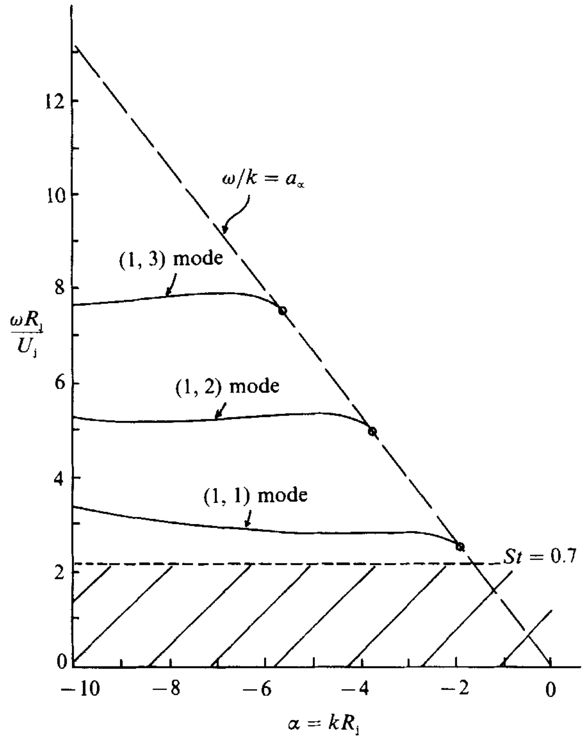


FIGURE 9. Wavenumber–frequency relations of helical neutral waves for cold jets; $M_j = 0.8$.

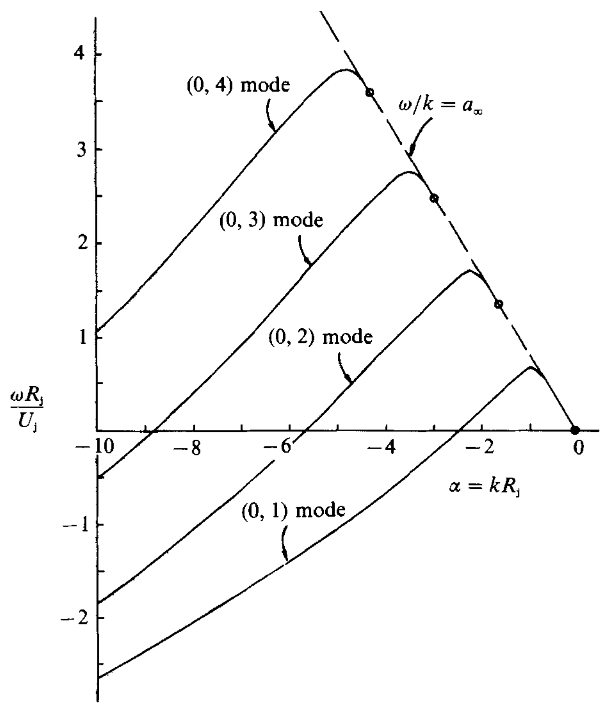


FIGURE 10. Wavenumber–frequency relations of supersonic axisymmetric neutral waves for cold jets; $M_j = 1.4$.

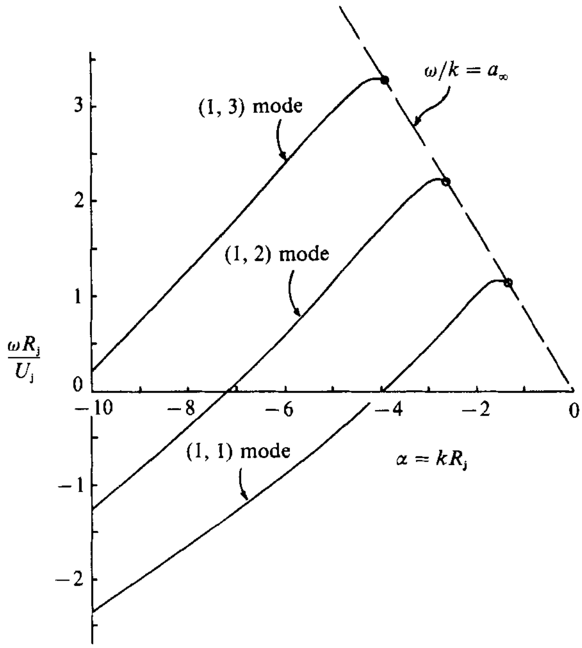


FIGURE 11. Wavenumber–frequency relations of supersonic helical neutral waves for cold jets; $M_j = 1.4$.

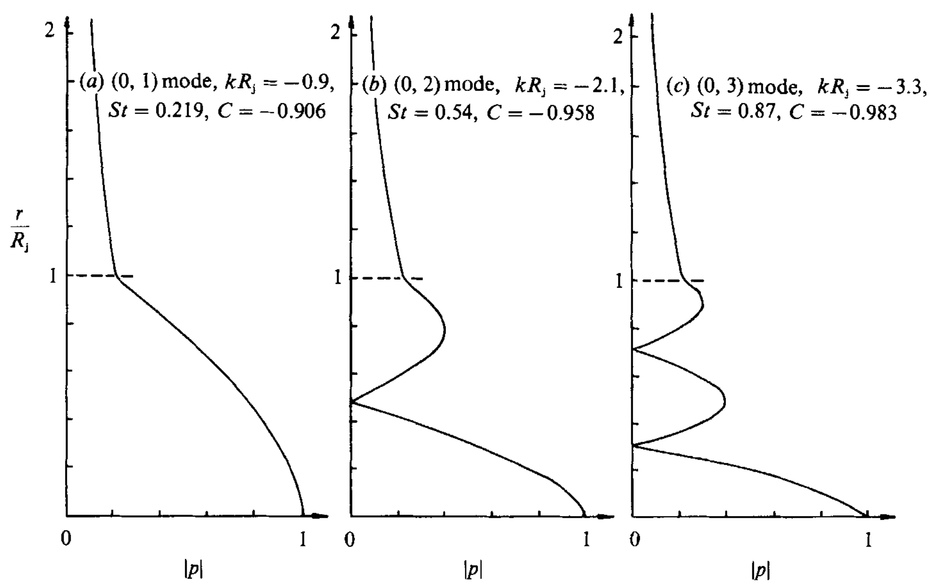


FIGURE 12. Eigenfunction distribution of supersonic axisymmetric neutral wave modes for cold jets; $M_j = 1.4$.

modes at $M_j = 1.4$. The eigenfunction extends over a long distance outside the jet. This means that the main part of the upstream-propagation wave is outside the jet. In a sense the wave is outside the jet but is guided by the jet on its way upstream. This appears reasonable and necessary since the jet flow is supersonic.

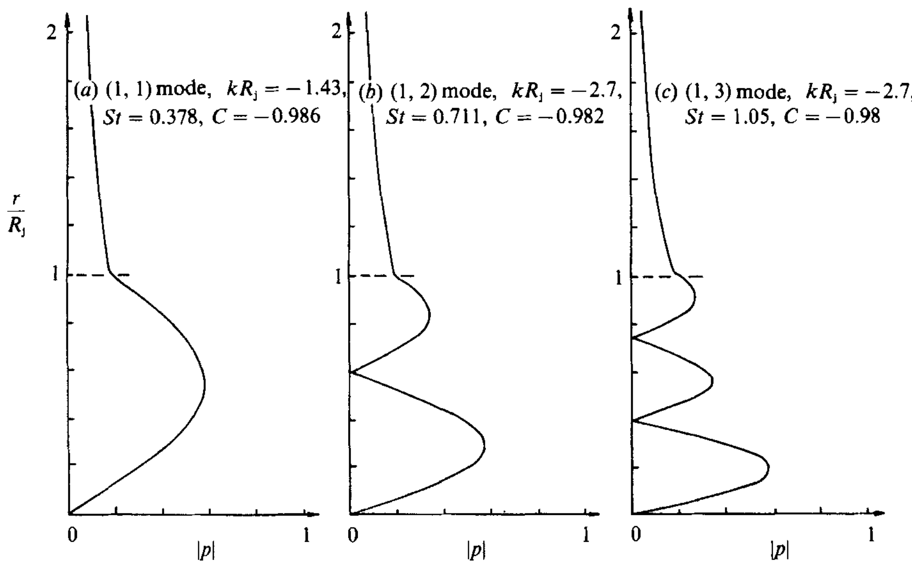


FIGURE 13. Eigenfunction distribution of supersonic helical neutral wave modes for cold jets; $M_j = 1.4$.

2.3. *Cut-off and minimum Strouhal numbers*

An examination of figures 8 and 9 and others at different subsonic Mach numbers indicates that the cut-off value of the Strouhal number (fD/U_j) as $C \to -1^+$ is essentially also the minimum Strouhal number of a particular (n, m) wave mode. The cut-off Strouhal numbers can be calculated by finding the roots of the dispersion function (16) in the limit $C \to -1^+$. For $n \neq 0$ (all non-axisymmetric modes) as $C \to -1^+$, (16) reduces to

$$2nJ_n(|\bar{\xi}_- \alpha|) + \frac{|\bar{\xi}_- \alpha|}{((a_\infty/a_j)+M_j)^2}[J_{n-1}(|\bar{\xi}_- \alpha|) - J_{n+1}(|\bar{\xi}_- \alpha|)] = 0, \tag{19}$$

where $\bar{\xi}_- = [((a_\infty/a_j)+M_j)^2-1]^{\frac{1}{2}}$. The roots of (19) can easily be found by Newton's iteration or similar methods.

For $n = 0$ (16) reduces to

$$\lim_{C\to -1^+} \frac{J_0(|\bar{\xi}_- \alpha|)}{|\alpha| \ln |\xi_+ \alpha|} + \frac{|\bar{\xi}_-|}{((a_\infty/a_j)+M_j)^2} J_1(|\bar{\xi}_- \alpha|) = 0; \tag{20}$$

because of the $|\alpha| \ln |\xi_+ \alpha|$ term, there are two types of solutions to (20).

(*a*) $C \to -1^+$, $\alpha \neq 0$

In this case (20) reduces to

$$J_1(|\bar{\xi}_- \alpha|) = 0. \tag{21}$$

This leads to $|\bar{\xi}_- \alpha| = \sigma_i$ where $J_1(\sigma_i) = 0$, $i = 1, 2, 3, \ldots$. Since $C \to -1^+$, this gives

$$\frac{\omega R_j}{U_j} \to -|\alpha| \frac{C}{M_j} \frac{a_\infty}{a_j} = \frac{\sigma_i}{M_j(a_j/a_\infty)[((a_\infty/a_j)+M_j)^2-1]^{\frac{1}{2}}}. \tag{22}$$

(*b*) $C \to -1^+$ and $|\alpha| \to 0$

In this case a special solution of (20) is possible. The solution has the form

$$|\alpha| \to 0^+, \quad C \to -1 + \beta\, e^{-\mu/|\alpha|^2} + \ldots, \tag{23}$$

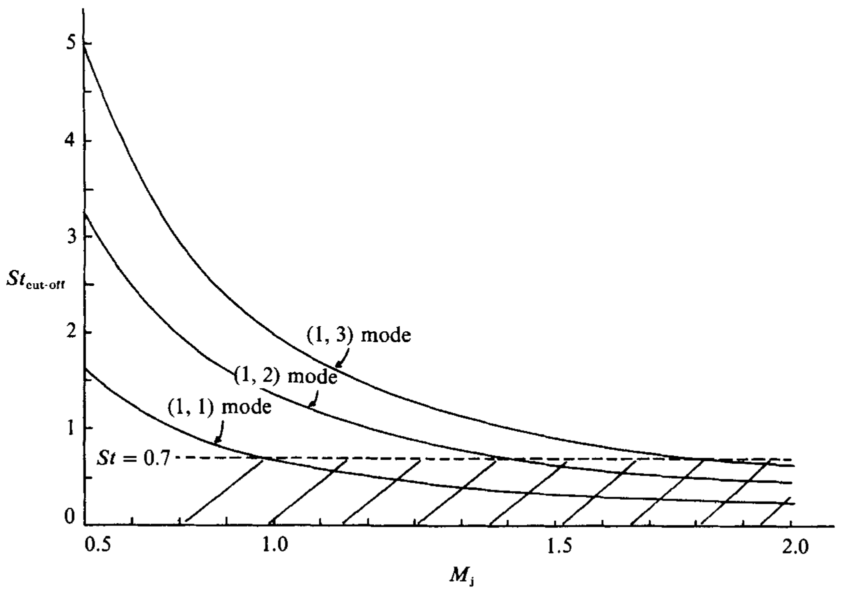


FIGURE 14. Cut-off (minimum) Strouhal number of the helical ($n = 1$) neutral waves as a function of jet Mach number for cold jets.

where $\beta (\beta > 0)$ and μ are unknown real constants. It is easy to show by substituting (23) into (20) that the equation is satisfied provided that

$$\mu = \frac{2((a_\infty/a_j)+M_j)^2}{|((a_\infty/a_j)+M_j)^2-1|}.$$

From (23) it is straightforward to find

$$\frac{\omega R_j}{U_j} \to 0^+ \quad \text{as} \quad |\alpha| \to 0. \tag{24}$$

Thus for the (0, 1) mode, the limiting value of the Strouhal number is zero or the dispersion function passes through the origin of the (ω, k)-plane.

Figure 14 shows the cut-off (minimum) Strouhal number of the helical modes (1, 1), (1, 2) and (1, 3) as a function of Mach number for cold jets. It is to be noted that for jet Mach number less than 0.7, the Strouhal numbers of these wave modes have values greater than unity. They decrease gradually as the jet Mach number increases into the supersonic region.

3. The feedback loop of impingement tones and its characteristics

In our proposed impingement tone generation model, the feedback loop is made up of downstream-propagating instability waves and the feedback upstream-propagating neutral acoustic waves. The overall characteristics of the loop are, therefore, dictated by those of the instability waves (Kelvin–Helmholtz) and the feedback neutral acoustic modes. It is well known that the instability waves in the core region of a jet are restricted to a fairly narrow frequency band. Experimental measurements and theoretical calculations indicate that the Strouhal number range is limited to less than 0.6 to 0.7. Instability waves of higher Strouhal number tend

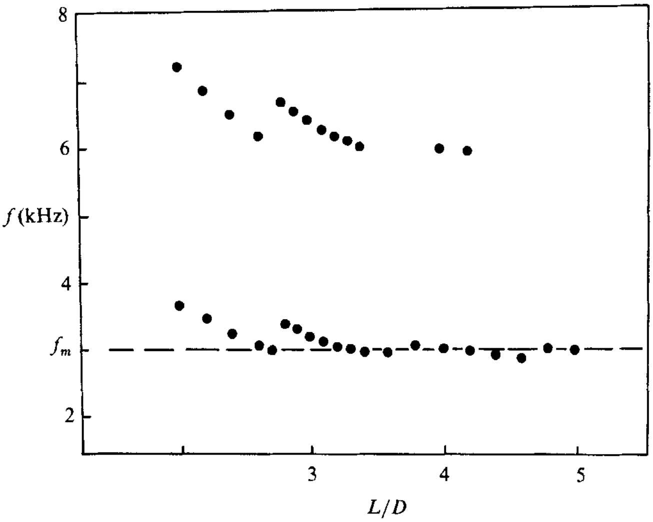


FIGURE 15. Data from Neuwerth (1974) showing ladder structure of impingement tone frequencies and average frequency, f_m. $M_j = 0.8$, $St_{av} = 0.568$, $D = 5$ cm.

to be damped out quickly and would be unable to propagate far enough downstream of the nozzle exit to reach the wall to excite the feedback acoustic modes unless the wall is placed extremely close to the jet exit. Since the feedback loop is driven by the instability waves of the jet flow, the Strouhal number of stable impingement tones must, therefore, be restricted to the range of less than 0.6 to 0.7.

Now if the feedback from the wall to the nozzle exit is accomplished by waves of the neutral acoustic modes discussed in the last section, it is clear that only those wave modes that exist in the range of Strouhal number less than 0.6 to 0.7 could participate. A careful examination of figures 8, 9 and 14 reveals that for cold subsonic jets only one neutral wave mode, namely the axisymmetric (0, 1) mode, lies in this restricted Strouhal number range. Thus, according to the present model the feedback phenomenon for subsonic impinging jets must possess axisymmetry. This theoretical deduction is consistent with all currently available experimental observations. For supersonic impinging jets, figure 14 indicates that the helical mode (1, 1) falls into the restricted Strouhal number range so that both axisymmetric and helical (or flapping) modes of resonant oscillations are possible. Both of these modes of resonance have been observed by Neuwerth (1974) and us (see figures 3 and 4).

The impingement tone frequency of a high subsonic jet of a fixed Mach number varies as the distance between the nozzle exit and the wall, L, gradually increases. A typical plot of frequency versus distance of separation is shown in figure 15. As can be seen the tone frequency as a function of L/D exhibits a ladder-like structure in accordance with (1). The discontinuities in frequency (also in the first harmonic) occur when there is a sudden increase in the integers n by unity. Neuwerth (1973, 1974), after a careful examination of his own data and those of Wagner (1971), recognized that the observed tone frequencies did not cover the full range of frequency given by (1). Instead he found that the range of Strouhal number of the tones was quite small. Because of this, it is meaningful to define an average Strouhal number (or average frequency, f_m) of the impingement tone (see figure 15). This average impingement tone Strouhal number is independent of the nozzle to wall separation distance. It is a function of Mach number alone. Being independent of

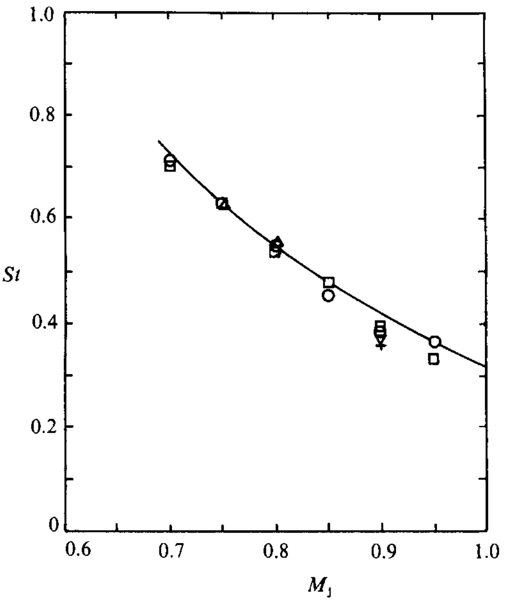


FIGURE 16. Comparison of the Strouhal number of the least dispersive (0, 1) mode neutral waves and the average Strouhal number of impingement tones: ——, least dispersive wave $\partial^2\omega/\partial k^2 = 0$; ○, Neuwerth (1972); □, Wagner (1971); △, Present study; ▽, Ho & Nosseir (1981); +, Umeda *et al.* (1987).

L/D, its value cannot be calculated by (1) which is based on the feedback loop phase integral condition. What mechanism determines this average Strouhal number? Here, it is proposed that this Strouhal number is controlled by the property of the feedback neutral acoustic wave. As shown in figures 8 and 9, the feedback neutral acoustic waves are dispersive waves. That is, different parts of a group of these waves will tend to propagate with different velocities and hence disperse in space in the course of time. In a stable feedback loop, the waves must form a coherent system against perturbations and fluctuations. This, therefore, favours waves which are the least dispersive or most coherent. In other words, the feedback loop is, in all likelihood, tuned to a frequency range near that of the least dispersive neutral acoustic wave. Thus to calculate the average impingement tone Strouhal number of a subsonic jet, it is only necessary to find the Strouhal number of the least dispersive (0, 1) mode neutral acoustic wave. The least dispersive wave is given by the condition

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} = 0. \tag{25}$$

Figure 16 shows the Strouhal number of the least dispersive (0, 1) mode wave as a function of (subsonic) jet Mach number. Plotted on this figure also are the average impingement tone Strouhal number measured by Neuwerth (1974), Wagner (1971), Ho & Nosseir (1981), Umeda *et al.* (1987), and those of the present study. As can be seen, over the Mach number range of 0.7 to 0.95 the agreement between the calculated and measured values is excellent. The excellent agreement between the calculated and the measured average Strouhal number appears to provide strong support for our proposal that the acoustic feedback from the wall to the nozzle exit is indeed accomplished by the neutral (0, 1) wave mode discussed in §2.

The vortex sheet jet model used in all the above calculations is, of course, only an

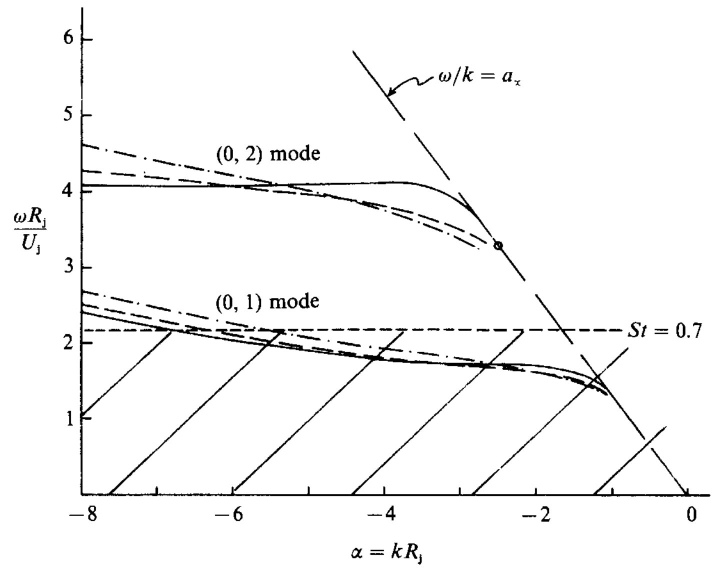


FIGURE 17. Effect of finite thickness mixing-layer on wavenumber–frequency relations of axisymmetric neutral waves for cold jets; $M_j = 0.8$: ——, vortex sheet jet model; – – –, $b/R_j = 0.5$; —·—·—, $b/R_j = 0.8$; hatched region $St = fD/U_j < 0.7$.

approximation. In a real jet the velocity profile is continuous and the mixing layer has a finite half-width. To assess the effect of finite mixing-layer thickness we have recomputed the dispersion relation using the same Gaussian velocity profile model as in the works of Tam & Burton (1984) and Tam & Hu (1989). The Gaussian velocity profile has been found to agree well with experimental measurements. The eigenvalue search procedure is identical to that described in Tam & Burton and, therefore, will not be repeated here. In the present case the numerical integration is simpler and straightforward since the waves propagate upstream so that there is no critical point. Figure 17 shows the calculated dispersion curves for a Mach 0.8 jet with $b/R_j = 0.0$, 0.5 and 0.8, where b is the half-width of the mixing layer and R_j is the radius of the jet. As can be seen, for the (0, 1) mode, which is the mode involved in the feedback loop, the dispersion relation is not much affected by the thickness of the jet mixing layer. For a nearly fully developed jet velocity profile with a mixing-layer half-width equal to half the jet radius ($b/R_j = 0.5$) the dispersion relation for the (0, 1) neutral wave mode differs from that of a vortex sheet jet by no more than 5 % in the relevant frequency range. This and similar calculations at other Mach numbers suggest that the vortex sheet model is a good first approximation.

The eigenfunction of the (0, 1) neutral wave mode at the average Strouhal number for subsonic jets has been calculated using (17) and (18). A typical distribution of the eigenfunction $|p|$ is shown in figure 6(a). In all the cases calculated, the pressure fluctuations are, for all intents and purposes, confined to regions inside the jet flow. This is in agreement with the findings of Wagner (1971) and Neuwerth (1974) who proposed that the feedback loop is completed by periodic acoustic disturbances propagating upstream inside the jet flow. Ho & Nosseir (1981) and Umeda *et al.* (1987) on the other hand have suggested that the feedback is accomplished by acoustic waves propagating upstream outside the jet. We believe that this is not the dominant feedback path. Most probably what they measured were the sound waves generated by the impingement of large-scale instability waves on the wall. These sound waves propagate in all directions away from the impingement region outside the jet. However, these sound waves are not part of the feedback loop as they are not the primary mechanism that causes the generation of instability waves by exciting the shear layer of the jet at the nozzle exit. The standing wave pattern observed by Wagner (1971) is formed by the superposition of the downstream-propagating instability wave and the upstream-propagating neutral wave. Such standing wave patterns were observed in the video records taken in the present study (but not presented here) as well.

To investigate whether the main feedback path is inside or outside the jet, Lepicovsky & Ahuja (1985) performed a high subsonic (Mach number = 0.79) impinging jet experiment inside an open wind tunnel. The open wind tunnel provided a uniform external flow surrounding the jet. In the experiment the wind tunnel Mach number could reach a value of 0.23. If the main feedback path is inside the jet the coaxial wind tunnel flow outside would affect the feedback loop minimally. But if the feedback path is outside the jet then when the wind tunnel is turned on the speed of the upstream-propagation sound wave will be reduced. This will, according to (1), decrease the frequency of the impingement tones. At a wind tunnel Mach number of 0.23 a substantial frequency drop should be observed. Lepicovsky & Ahuja, however, found little frequency shift over the entire range of wind tunnel operating Mach numbers. Their experiment thus offers additional evidence in support of the original proposal of Wagner (1971) and Neuwerth (1974).

It is worthwhile to point out that the calculated and predicted average impingement tone Strouhal number increases rapidly as the jet Mach number decreases (see figure 16). For jet Mach number less than 0.65 the predicted average Strouhal number exceeds the unstable Strouhal number range of the jet, namely 0.7 or less. Thus for lower Mach number jets there will be a mismatch in the Strouhal number between the downstream-propagating instability waves and the least dispersive feedback upstream-propagating acoustic waves. Under this circumstance it is very unlikely that the feedback loop can be self-sustained. Hence stable impingement tones would not be emitted. That no stable tones would be emitted by a high Reynolds number subsonic impinging jet with Mach number less than, say, 0.6 has been well documented since the pioneering experiments of Wagner (1971). The above, however, appears to provide the first concrete theoretical explanation of this low Mach number cut-off phenomenon.

The present work is confined to high Reynolds number ($Re > 2\times10^5$) subsonic round impinging jets. It is well known that low Reynolds number jets and mixing layers are extremely sensitive to acoustic excitation. Because of their high receptivity to external excitation they can emit tones by other weaker feedback paths at Mach numbers well below 0.6 (see Karamcheti *et al.* 1969), when the present proposed feedback mechanism is not effective. In addition to the Reynolds number effect it is also important to point out that the geometry of the jet is also a factor that must be taken into consideration. The characteristics of the Kelvin–Helmholtz instability waves which drive the feedback loop and those of the upstream-propagating neutral acoustic waves are flow geometry dependent. One simple way to see this is to note that a circular jet can support helical wave motion as well as flapping motion but a two-dimensional jet or an elliptic jet can sustain only flapping but not helical motion. For non-circular high Reynolds number high subsonic impinging jets we believe that a similar feedback loop as discussed in this paper exists. However, the characteristics of the impingement tones would invariably be modified by the non-circular geometry of the jet.

Here we would like to make it clear that the proposed feedback cycle of this paper is not the only one possible. What we suggest is that if we accept the ansatz that it is the dominant feedback mechanism for high Reynolds number impinging jets then it is possible to explain: (i) why only axisymmetric instability wave modes are observed for subsonic jets whereas both axisymmetric and helical (or flapping) modes are observed for supersonic jets, (ii) why there is a low Mach number cut-off of the impingement tone phenomenon around Mach number 0.6. In addition, our feedback model offers, for the first time, a way to predict the average impingement tone frequencies. The calculated results are found to agree well with experimental measurements. We have by no means proven that the ansatz is correct. However, we believe that the good agreement between calculated results and measurements does provide strong support for its validity.

4. Summary

It is proposed that the feedback cycle of the impingement tones is accomplished by the intrinsic neutral acoustic waves of the jet flow. The characteristics of these neutral waves have been carefully examined. It is found that for subsonic jets only the (0, 1) mode of the neutral acoustic waves has Strouhal numbers which match those of the Kelvin–Helmholtz instability wave of the jet flow. Because of this, only axisymmetric feedback resonance is possible for high Reynolds number subsonic impinging jets. In contrast to this, for supersonic jets both the axisymmetric (0, 1) and helical (1, 1) modes satisfy the frequency matching condition and thus are possible modes of resonance. The theoretical result is consistent with experimental observations.

The Strouhal number range of the impingement tones of a subsonic jet at a fixed Mach number is quite small. This allows one to define, in a meaningful way, an average Strouhal number of the tones. It is proposed that this average Strouhal number must be equal to that of the least dispersive (0, 1) mode neutral wave. Excellent agreement between the calculated Strouhal number of the least dispersive wave and the average Strouhal number measured by Neuwerth (1974) and Wagner (1971), Ho & Nosseir (1981) and Umeda *et al.* (1987) and those of the present study is found. The eigenfunctions, $|p|$, of the neutral feedback waves at the average Strouhal numbers are found to be effectively confined to the inside of the jet column. This finding is in agreement with the feedback concept of Wagner who proposed that the feedback acoustic disturbances travelled upstream inside the jet flow.

Present numerical results indicate that for subsonic (cold) jets with Mach number less than 0.65 the Strouhal numbers of the least dispersive (0, 1) mode upstream-propagating neutral acoustic waves are larger than 0.7. They are, therefore, outside the Strouhal number range of the Kelvin–Helmholtz instability waves of the jet. This mismatch in Strouhal number suggests that no stable feedback loop could be maintained and hence no stable impingement tone would be emitted by high Reynolds number subsonic jets. This theoretical finding appears to provide a much sought after explanation of the rather puzzling low Mach number cut-off phenomenon reported by most impingement tone experiments since the early work of Wagner (1971).

The present study is limited in objectives. Issues involving broadband impinging noise as well as the directivity of impingement tones, etc., are beyond the scope of the present investigation.

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